by revolving BD around OZ is the same as that generated by revolving the hyperbola (2) around OZ, its conjugate axis; that is, the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{c^2}=1.$$

If the moving line is perpendicular to the fixed line, it will lie wholly in the plane $Z=k$, and the hyperboloid will degenerate into the plane figure $Z=k$, $x^2+y^2 \mathrel{\overline{>}} a^2$, a in this case being the shortest distance between the lines OZ and BD.

THE SOLUTION OF AN EQUATION BY A FRAME.

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The *frame* of the equation

$$f(x)=x^n+bx^{n-1}+cx^{n-2}+dx^{n-3}+\ldots=0,$$

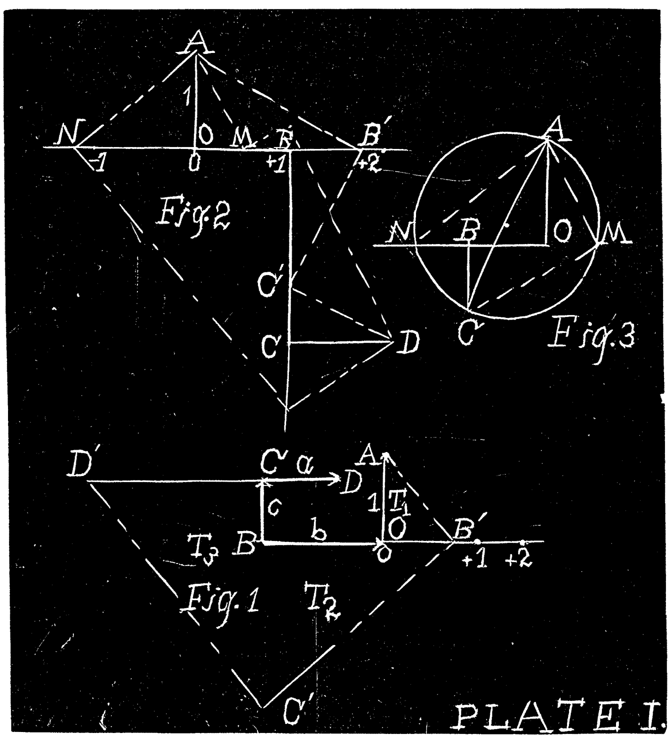

PLATE I. $OB'=r$, $C'B=r^2+br$, $D'C=r^2+br^2+c$ in Fig. 1. Also, $b'=r+b=BB'$, $c'=r^2+br+c=C'C$. In figure after 1, b, c, ... are omitted as their use is evident from Fig. 1.

(see Fig. 1) consists of a succession of strokes, AO, OB, BC, CD, ..., successive strokes being at right angles, $AO=a=-1$, $OB=-b$, $BC=c$, $CD=d$, $DE=-e$, $EF=-f$, $FG=-g$, ...

To substitute r for x, OB' representing r, draw the *path* AB', $B'C'$, $C'D'$, ..., T_1, T_2, T_3, ... These are easily seen to be similar, and as OB' is r times OA, BC is r times BB', ... Hence the values on the figure. If $f(x)=x^3+bx^2+cx+d$, $f(r)=D'D$. If r is a root of $f(x)=0$, D' falls at D.

To draw the path for $x=r$, place one edge of a rectangular card through A and a point near B', place another card against it and a third against the second and so the proper edge passes through D. Now A and D remaining fixed, slide the cards to locate B' and C'. A little practice makes this easy. Fig. 2 solves $x^3-x^2-2x+1=0$. Horners method will give as many figures as wished. After locating one root, *e. g.*, that near $+2$, we may draw, Fig. 3, the frame of the quadratic having the other two roots. In this b and c are the b' and c' of Fig. 2. Just so in the numerical way,

$$\begin{array}{lccccccc|l} & 1 & + & b & + & c & + & d & \underline{r} \\ & & & r & & r^2+br & & & \\ \hline x^2+b'x+c'=0 \qquad & 1, & & r+b=b', & & r^2+br+c=c', & & d'=0 & \end{array}$$

As the Runge Circle cuts the ray of b the other two roots are real. They should check with those of Fig. 2.

As a second illustration we will take the cubic $x^3-4x^2+9x-10=0$. Fig. 4 gives one real root. As the Runge Circle (*i. e.*, the circle on the diameter AC, Fig. 5) does not cut the ray of b, the other two roots are complex.

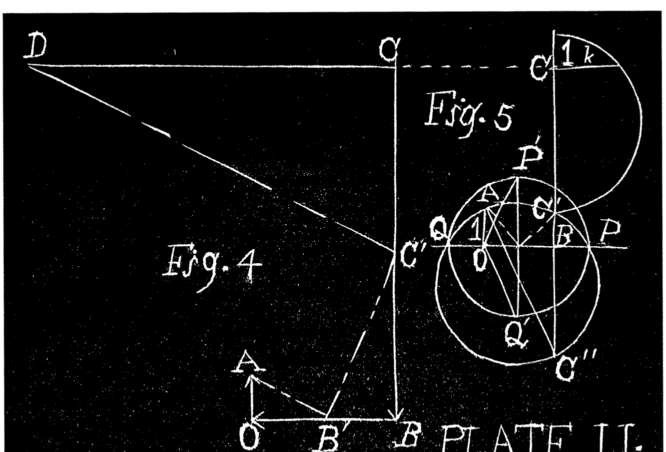

PLATE II. As $OB=b$ (numerically)$=2h$, it is evident $BC'=h^2$.

Of the four methods that I have devised for the solution of a quadratic with complex roots I give two. Of these, II seems the "slicker." To consider all of them in their relations would alone be enough for a paper.

Starting with the frame $AOBC$ (Fig. 5), b and c being the b' and c' of Fig. 4, we draw the path $AB'C'$ for *equal roots.* If the roots are $h+ki$ and $h-ki$, $h=-\frac{1}{2}b$, $h^2+k^2=c$. Therefore, $C'C=c-h^2=k^2$.

I. Find k as in upper part of Fig. 5. II. Make $C'C''=C'C$, and solve the frame $AOBC''$ giving the root-points P and Q. Then P' and Q' are the root-points of the given quadratic frame $AOBC$, since $y^2-2hy=k^2-h^2$ gives $y=h\pm k$.

The *linkage* of $f(x)$ for a given complex value of x is the succession of strokes, or links, which starting from the origin has the terminus of one link as the origin of the next, and each link represents the corresponding term of $f(x)$. If the value of x used is a root of $f(x)$ the last link terminates at the origin. The point representing x, or "point x," is the root-point. The point X representing $f(x)$, or "point X," is the function-point. As we may solve $x^n=N$ by a frame, *e. g.*, $x^3+0x^2+0x-N=0$, Fig. 6 gives an easy way of finding the lengths of x^2, x^3, x^4, ... Their slants being two, three, four, ... times that of x. Fig. 7 gives the linkages for $x^3-4x^2+9x-10=0$

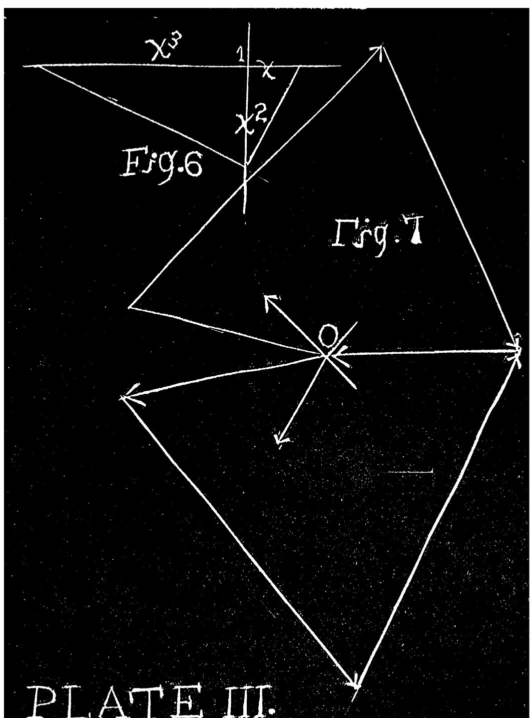

PLATE III. The successive strokes in Fig. 7 from O to O again are the successive terms of the polynomial $x^3-4x^2+9x-10$. Those in the upper part of the figure (light strokes) are for $x=x_2$. Those in the lower part of the figure (heavy strokes) are for $x=x_1$.

for $x_1=OP$, $x_2=OQ$, the unit being two-fifths that of Fig. 5. It is easy to show that the termini of corresponding links are conjugate points symmetrical as to the axis of reals. Hence if X_1 falls at the origin, so does X_2. We easily have a graphical proof of the theorem. Complex roots enter $f(x)=0$ in conjugate pairs.

Let us now consider a more general method in which all roots are regarded as of the form $h+ki$, where either h or k may be zero.

The chief interest in the method is its generality. The "double position" method used is: Having found R_1 and R_2, the function-points corresponding to r_1 and r_2, construct on r_1r_2 a triangle $r_1r_2 0$ similar to R_1R_2O, and take 0 as a first approximation to the root-point of the origin O. In Figs. 8, 9, and 10, the method is applied to $x^4-5x^3+13x^2-19x+10=0$.

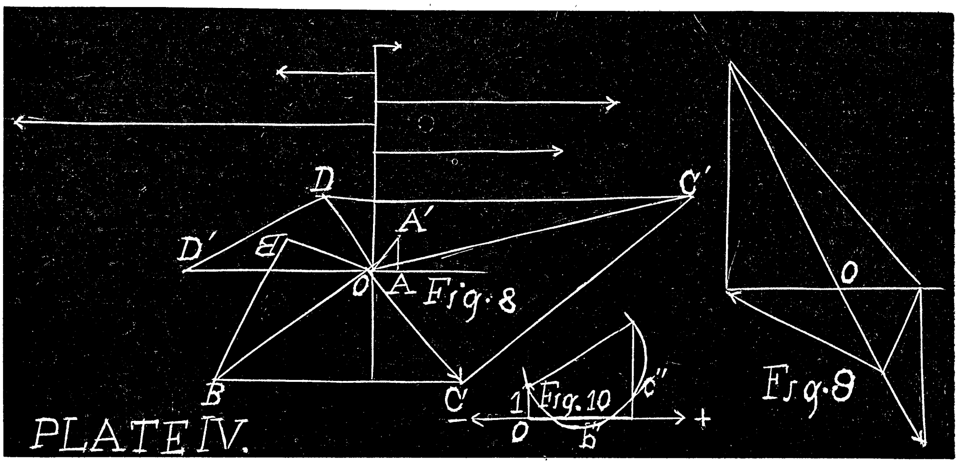

PLATE IV. Referring to Fig. 1 and the note concerning it, it is seen that, if r be for r_1, $OA'=r$, $A'B=b$. $\therefore OB=b'$, $OB'=b'r$, $B'C=c$. $\therefore OC=c'$, $OC'=c'r$, $C'D=d$. $\therefore OD=d'$ and $OD'=d'r$. As $OD'+e=0$. Thus is the last stroke terminate at O, r is a root.

It is to be remembered that, "Geometric multiplication consists in doing with the multiplicand (as regards slant and slide, *i. e.*, turning and stretching) what must be done to the initial unit, (1_0), to obtain the multiplier;" also that, "The geometric sum of two consecutive strokes is the stroke from the first origin to the last terminus." In Fig. 8, *for the multiplication* the triangles OAA', OBB', OCC' and ODD' are similar. *For the addition*, $OB=OA'+A'B$ or $r+b=b'$, $OC=OB'+B'C=b'r+c=r^2+br+c=c'-\ldots$ In (9) a', b', c', d' of (8) are used to find a'', b'', c''. These are all reals. Thus in (10) we have the ordinary frame for the quadratic with roots 1 and 2. (Unit in (9) and (10) twice that in (8).) Thus the roots of $x^4-5x^3+13x^2-19x+10=0$ are $1+2i$, $1-2i$, $+1$, and $+2$.

It seems to me that, if we carry out what is touched upon in connection with Fig. 5, it will be seen that it is more logical to regard the roots as $1\pm 2i$, where $k=2$ and $\frac{3}{2}\pm(\frac{1}{2}i)i$ where $k=\frac{1}{2}i$ than to say they are $1+2i$, $1-2i$, $1+0.i$, and $2+0.i$.